Qualitative analysis of solutions for a class of logarithmic Kirchhoff equation with distributed delay

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Abstract

In this article, we concerned with a logarithmic Kirchhoff equation with distributed internal delay. Firstly, we obtain the global existence of solutions by using the well-depth method. Later, under appropriate assumptions on the weight of the delay and that of frictional damping, we establish the exponential decay. Moreover, we obtain the blow up results for negative initial energy.

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1 Introduction

In this article, we deal with the logarithmic Kirchhoff equation with distributed delay as follows:

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \mu_1 u_t(x,t) + \int_{\tau_1}^{\tau_2} \mu_2(s)\, u_t(x,t-s)\, ds & \\ = u|u|^{p-2}\ln|u|^k, & x \in \Omega,\ t>0, \\ u(x,t) = 0, & x \in \partial\Omega, \\ u_t(x,-t) = f_0(x,t), & \text{in } (0,\tau_2), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of R^n $(n \geq 1)$ with a smooth boundary $\partial\Omega$. k, μ_1 are positive constants, the integral term denotes the distributed delay for $0 \leq \tau_1 < \tau_2$ and $\mu_2 : [\tau_1, \tau_2] \to [0,\infty)$ is a bounded function. u_0, u_1, f_0 are the initial data functions to be specified later. $M(s)$ is a nonnegative function of C^1 for $s \geq 0$ satisfies, $M(s) = m_0 + \alpha s^\gamma$, $m_0 > 0$, $\alpha \geq 0$ and $\gamma \geq 0$, specially, we take $M(s) = 1 + s^\gamma$ where $m_0 = 1$, $\alpha = 1$ for the problem (1.1).

Generally, time delays appear in many practical problems such as thermal, economic phenomena, biological, chemical, physical, electrical engineering systems, mechanical applications and medicine [14]. Logarithmic nonlinearity often seems in nuclear physics, inflation cosmology, geophysics and optics (see [3, 8]). The Kirchhoff equation is among the famous wave equation's model which describe small vibration amplitude of elastic strings. It has been introduced in 1876 by Kirchhoff [13]. To be more precise, Kirchhoff recommended a model denoted by the equation for $f = g = 0$,

$$\rho h \frac{\partial^2 u}{\partial t^2} + \delta \frac{\partial u}{\partial t} + g\left(\frac{\partial u}{\partial t}\right) = \left\{\rho_0 + \frac{Eh}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right\}\frac{\partial^2 u}{\partial x^2} + f(u), \tag{1.2}$$

for $0 < x < L$, $t \geq 0$, here $u(x,t)$ is the lateral displacement, E is the Young modulus, ρ is the mass density, h is the cross-section area, L is the lenght, ρ_0 is the initial axial tension, δ is the resistance

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modulus, and f and g are the external forces. Furthermore, (1.2) is called a degenerate equation when $\rho_0 = 0$ and nondegenerate one when $\rho_0 > 0$.

Firstly, for the literature review, we begin with the works of Birula and Mycielski [4, 5]. They studied the following equation with logarithmic term:

$$u_{tt} - u_{xx} + u - \varepsilon u \ln |u|^2 = 0. \tag{1.3}$$

They are the pioneer of these kind of problems. They established that, in any number of dimensions, wave equations with the logarithmic term have localized, stable, soliton-like solutions. Cazenave and Haraux [6], in 1980, studied the equation as follows:

$$u_{tt} - \Delta u = u \ln |u|^k . \tag{1.4}$$

The authors obtained the existence and uniqueness of the equation (1.4). Gorka [8] obtained the global existence for one-dimensional of the equation (1.4). Bartkowski and Gorka [3], studied the weak solutions and the existence results.

In [10], Hiramatsu et al. concerned with the equation as follows:

$$u_{tt} - \Delta u + u + u_t + \left|u^2\right| u = u \ln u. \tag{1.5}$$

In [9], Han obtained the global existence results for the equation (1.5).

In 1986, Datko et al. [7], proved that, a small delay effect is a source of instability. In [16], Nicaise and Pignotti studied the following equation:

$$u_{tt} - \Delta u + a_0 u_t (x,t) + a u_t (x, t-\tau) = 0, \tag{1.6}$$

here a_0, $a > 0$. They proved that, under the condition $0 \leq a \leq a_0$, the system is exponentially stable. They proved a sequence of delays that shows the solution is instable for $a \geq a_0$. Without delay, some other authors [15, 37] looked into exponentially stability of the equation (1.6).

In [18], Nicaise et al. studied the wave equation in one space dimension with time-varying delay. In that work, the authors showed that the exponential stability with the condition

$$a \leq \sqrt{1-d} a_0,$$

here d is a constant and

$$\tau' (t) \leq d < 1, \forall t > 0.$$

In [17], Nicaise and Pignotti introduced the distributed delay:

$$\int_{\tau_1}^{\tau_2} \mu_2 (s) u_t (x, t-s) ds. \tag{1.7}$$

Under appropriate conditions, they established the exponential stability of the solution with disributed delay.

In [12], Kafini and Messaoudi concerned with the following wave equation with delay and logarithmic terms:

$$u_{tt} - \Delta u + \mu_1 u_t (x,t) + \mu_2 u_t (x, t-\tau) = |u|^{p-2} u \log |u|^k . \tag{1.8}$$

They established the local existence and blow up of solutions for the equation (1.8).

Wu and Tsai [35], considered the following Kirchhoff-type equation:

$$u_{tt} - M\left(\|\nabla u\|_2^2\right)\Delta u + |u_t|^{r-2} u_t = |u|^{p-2} u, \tag{1.9}$$

they obtained the blow up results of the equation (1.9). In 2013, Ye [36], studied the global existence by constructing a stable set in $H_0^1(\Omega)$ and proved the decay of solutions by utilizing a lemma of Komornik for the nonlinear Kirchhoff-type equation (1.9) with dissipative term.

When $M(s) \equiv 1$, in [11], Kafini considered the following wave equation with logarithmic nonlinearity with distributed delay:

$$u_{tt} - \Delta u + \mu_1 u_t(x,t) + \int_{\tau_1}^{\tau_2} \mu_2(s)\, u_t(x,t-s) = u|u|^{p-2}\ln|u|^k, \tag{1.10}$$

he established the local and global existence. Moreover, he proved the exponential decay of the equation (1.10).

In [19], Park concerned with the following equation with delay and logarithmic terms:

$$u_{tt} - \Delta u + \alpha u_t(t) + \beta u_t(x,t-\tau) = u\ln|u|^\gamma. \tag{1.11}$$

The author established the local and global existence of solutions for the equation (1.11). Also, the author proved the decay and nonexistence of solutions of the equation (1.11). In recent years, some other authors investigate hyperbolic type equations (see [2, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34]).

In this paper, we considered the global existence, exponential decay and blow up of solutions for the logarithmic Kirchhoff equation (1.1) with distributed delay, motivated by above works. To our best knowledge, there is no research, related to the logarithmic Kirchhoff equation (2.2) with distributed delay term ($\int_{\tau_1}^{\tau_2} \mu_2(s)\, u_t(x,t-s)\,ds$) and logarithmic source term ($u|u|^{p-2}\ln|u|^k$), hence, our work is the generalization of the above studies.

This work consists of five sections in addition to the introduction: Firstly, in Section 2, we give some needed materials. Then, in Section 3, we get the global existence results by the well-depth method. Moreover, in Section 4, we obtain the exponential decay. Finally, in Section 5, we establish the blow up results for negative initial energy.

2 Preliminaries

In this section, we give some materials for our main result. As usual, the notation $\|.\|_p$ denotes L^p norm, and $(.,.)$ is the L^2 inner product. In particular, we write $\|.\|$ instead of $\|.\|_2$.

Let $B_p > 0$ be the constant satisfying [1]

$$\|v\|_p \le B_p \|\nabla v\|_p, \text{ for } v \in H_0^1(\Omega). \tag{2.1}$$

Similar to the [16], we introduce the new variable

$$z(x,\rho,s,t) = u_t(x,t-\rho s) \text{ in } \Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

Therefore, we obtain

$$s z_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0 \text{ in } \Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

Hence, the problem (1.1) is equivalent to:

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2\right)\Delta u + \mu_1 u_t(x,t) & \\ + \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)\,ds = u|u|^{p-2}\ln|u|^k & \text{in } \Omega\times(\tau_1,\tau_2)\times(0,\infty) \\ s z_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0 & \text{in } \Omega\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty) \\ z(x,\rho,s,0) = f_0(x,-\rho s) & \text{in } \Omega\times(0,1)\times(\tau_1,\tau_2) \\ u(x,t) = 0 & \text{on } \partial\Omega\times(0,\infty) \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{in } \Omega. \end{cases} \tag{2.2}$$

The energy functional related to the problem (2.2) is, for $\forall t \geq 0$,

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \frac{k}{p^2}\|u\|_p^p \\ &\quad + \frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(\xi+\mu_2(s))|z(x,\rho,s,t)|^2\,ds\,d\rho\,dx \\ &\quad - \frac{1}{p}\int_\Omega |u|^p \ln|u|^k\,dx, \end{aligned} \tag{2.3}$$

where ξ is a positive constant satisfying

$$\mu_1 > \int_{\tau_1}^{\tau_2}\mu_2(s)\,ds + \frac{\xi}{2}(\tau_2-\tau_1), \tag{2.4}$$

under the condition

$$\mu_1 > \int_{\tau_1}^{\tau_2}\mu_2(s)\,ds.$$

The following lemma shows that the related energy functional is nonincreasing.

Lemma 2.1. Suppose that (2.4) holds. Then, along the solution of (2.2) and for some $C_0 \geq 0$, we get

$$E'(t) \leq -C_0\int_\Omega\left(|u_t|^2 + |z(x,1,s,t)|^2\right)dx \leq 0. \tag{2.5}$$

Proof. By multiplying the first equation in (2.2) by u_t and integrating over Ω and the second equation in (2.2) by $(\xi+\mu_2(s))z$ and integrating over $(\tau_1,\tau_2)\times(0,1)\times\Omega$ with respect to s, ρ and x, summing up, we obtain

$$\begin{aligned} &\frac{d}{dt}\left(\begin{array}{c} \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \frac{k}{p^2}\|u\|_p^p - \frac{1}{p}\int_\Omega |u|^p\ln|u|^k\,dx \\ + \frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(\xi+\mu_2(s))|z(x,\rho,s,t)|^2\,ds\,d\rho\,dx \end{array}\right) \\ =&\ -\mu_1\int_\Omega |u_t|^2\,dx - \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}(\xi+\mu_2(s))zz_\rho(x,\rho,s,t)\,ds\,d\rho\,dx \\ &\ -\int_\Omega u_t\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,s,t)\,ds\,dx. \end{aligned} \tag{2.6}$$

Now, we consider the last two terms of the right-hand side of (2.6) as:

$$\begin{aligned} & -\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} (\xi + \mu_2(s)) z z_\rho (x,\rho,s,t)\, ds d\rho dx \\ = & -\frac{1}{2} \int_\Omega \int_{\tau_1}^{\tau_2} \int_0^1 \frac{\partial}{\partial \rho} \left[(\xi + \mu_2(s)) |z(x,\rho,s,t)|^2 \right] d\rho ds dx \\ = & \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} \mu_2(s)\, ds + \xi (\tau_2 - \tau_1) \right) \int_\Omega |u_t|^2 dx \\ & -\frac{1}{2} \int_\Omega \int_{\tau_1}^{\tau_2} (\xi + \mu_2(s)) |z(x,1,s,t)|^2 ds dx \end{aligned}$$

and

$$\begin{aligned} & -\int_\Omega u_t \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)\, ds dx \\ \leq & \frac{1}{2} \left(\int_{\tau_1}^{\tau_2} \mu_2(s)\, ds \int_\Omega |u_t|^2 dx + \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds \int_\Omega |z(x,1,s,t)|^2 dx \right). \end{aligned}$$

Therefore, we get

$$\begin{aligned} \frac{dE(t)}{dt} \leq & -\left(\mu_1 - \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds - \frac{\xi}{2} (\tau_2 - \tau_1) \right) \int_\Omega |u_t|^2 dx \\ & -\frac{\xi}{2} \int_\Omega \int_{\tau_1}^{\tau_2} |z(x,1,s,t)|^2 ds dx. \end{aligned}$$

By using (2.4), we obtain, for some $C_0 > 0$,

$$E'(t) \leq -C_0 \int_\Omega \left(|u_t|^2 + \int_{\tau_1}^{\tau_2} |z(x,1,s,t)|^2 ds \right) dx \leq 0.$$

Q.E.D.

3 Global existence

In this section, we establish that the solution of (2.2) is uniformly bounded and global in time. For this aim, we set

$$I(t) = \|\nabla u\|^2 - \int_\Omega |u|^p \ln |u|^k dx,$$

$$\begin{aligned} J(t) = & \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2(\gamma+1)} \|\nabla u\|^{2(\gamma+1)} + \frac{k}{p^2} \|u\|_p^p \\ & + \frac{1}{2} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s (\xi + \mu_2(s)) z^2 ds d\rho dx - \frac{1}{p} \int_\Omega |u|^p \ln |u|^k dx. \end{aligned} \quad (3.1)$$

Therefore,

$$E(t) = J(t) + \frac{1}{2} \|u_t\|^2.$$

Lemma 3.1. Assume that the initial data $u_0,\, u_1 \in H_0^1(\Omega) \times L^2(\Omega)$ satisfying

$$I(0) > 0 \text{ and } \beta = kC_{p+l}\left(\frac{2pE(0)}{p-2}\right)^{\frac{p-2+l}{2}} < 1. \tag{3.2}$$

Then, $I(t) > 0$, for any $t \in [0, T]$.

Proof. Since $I(0) > 0$ we infer by continuity that there exists $T^* \leq T$ such that $I(t) \geq 0$ for all $t \in [0, T^*]$. This implies that, for all $t \in [0, T^*]$,

$$\begin{aligned} J(t) &= \frac{p-2}{2p}\|\nabla u\|^2 + \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \frac{k}{p^2}\|u\|_p^p \\ &+ \frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(\xi + \mu_2(s))\, z^2 ds d\rho dx + \frac{1}{p}I(t), \end{aligned}$$

yields

$$J(t) \geq \frac{p-2}{2p}\|\nabla u\|^2.$$

Hence,

$$\|\nabla u\|^2 \leq \frac{2p}{p-2}J(t) \leq \frac{2p}{p-2}E(t) \leq \frac{2p}{p-2}E(0). \tag{3.3}$$

On the other hand, by using the fact that $\ln|u| < |u|^l$, we get

$$\int_\Omega |u|^p \ln|u|\, dx \leq \int_\Omega |u|^{p+l} dx, \tag{3.4}$$

where l is choosen to be $0 < l < \frac{2}{n-2}$, such that

$$p + l < \frac{2n-2}{n-2} + l < \frac{2n}{n-2}.$$

Therefore, the embedding $H_0^1(\Omega) \hookrightarrow L^{p+l}(\Omega)$, satisfies

$$\begin{aligned} \int_\Omega |u|^p \ln|u|\, dx &\leq C_{p+l}\|\nabla u\|^{p+l} = C_{p+l}\|\nabla u\|^2 \|\nabla u\|^{p-2+l} \\ &= C_{p+l}\|\nabla u\|^2 \left(\|\nabla u\|^2\right)^{\frac{p-2+l}{2}} \\ &\leq C_{p+l}\left(\frac{2pE(0)}{p-2}\right)^{\frac{p-2+l}{2}}\|\nabla u\|^2, \end{aligned} \tag{3.5}$$

where C_{p+l} is the embedding constant.

As a result, by (3.1) and (3.2), we infer that

$$I(t) > \|\nabla u\|^2 - \beta\|\nabla u\|^2 > 0,\ \forall t \in [0, T^*]. \tag{3.6}$$

By repeating this procedure, T^* can be extended to T. Q.E.D.

Theorem 3.2. If the initial data u_0, u_1 satisfy the conditions of Lemma 3.1, then the solution of (2.2) is uniformly bounded and global in time.

Proof. It suffices to show that $\|\nabla u\|^2 + \|u_t\|^2$ is bounded independently of t. We see that,

$$\begin{aligned} E(0) &\geq E(t) = \frac{1}{2}\|u_t\|^2 + J(t) \\ &\geq \frac{1}{2}\|u_t\|^2 + \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \frac{k}{p^2}\|u\|_p^p \\ &\quad + \frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(\xi+\mu_2(s))\, z^2 ds d\rho dx + \frac{1}{p}I(t) \\ &\geq \frac{1}{2}\|u_t\|^2 + \frac{1}{p}(1-\beta)\|\nabla u\|^2. \end{aligned}$$

Thus,

$$\|\nabla u\|^2 + \|u_t\|^2 \leq CE(0),$$

here C is a positive constant depending only on k, p and C_{p+1}. Q.E.D.

4 Exponential decay

In this section, we prove our main decay result. Firstly, we give the following lemmas:

Lemma 4.1. [11] The functional

$$F_1(t) = \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} se^{-\rho s}(\xi+\mu_2(s))\,|z(x,\rho,s,t)|^2 ds d\rho dx$$

satisfies, along the solution of (2.2), for some c_1, $c_2 > 0$,

$$F_1'(t) \leq c_1\|u_t\|^2 - c_2\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(\xi+\mu_2(s))\,|z(x,\rho,s,t)|^2 ds d\rho dx. \tag{4.1}$$

Lemma 4.2. The functional

$$F_2(t) = NE(t) + \varepsilon\int_\Omega uu_t dx + \frac{\varepsilon\mu_1}{2}\int_\Omega |u|^2 dx$$

satisfies, along the solution of (2.2)

$$\begin{aligned} F_2'(t) \leq\; & -(NC_0-\varepsilon)\|u_t\|^2 - \varepsilon(1-\beta-\delta)\|\nabla u\|^2 \\ & -\varepsilon\|\nabla u\|^{2(\gamma+1)} - \left(NC_0 - \varepsilon\frac{c_*}{4\delta}\right)\int_\Omega\int_{\tau_1}^{\tau_2} z^2(x,1,s,t)\, ds dx, \end{aligned} \tag{4.2}$$

here N, α and ε are positive constants.

Proof. Differentiation, by using equations in (2.2), satisfies

$$\begin{aligned} F_2'(t) &\leq -NC_0 \int_\Omega \left(|u_t|^2 + |z(x,1,s,t)|^2 \right) dx \\ &+\varepsilon \left(\int_\Omega |u_t|^2 dx - \int_\Omega |\nabla u|^2 dx + \int_\Omega |u|^p \ln |u|^k dx \right) \\ &-\varepsilon \int_\Omega \|\nabla u\|^{2\gamma} |\nabla u|^2 dx - \varepsilon \int_\Omega u \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t) ds dx. \end{aligned} \tag{4.3}$$

Utilizing Young's inequality and the boundness property of $\mu_2(s)$, we obtain, for any $\delta > 0$ and some $c_* > 0$,

$$\begin{aligned} &-\int_\Omega u \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t) ds dx \\ &\leq \delta \|\nabla u\|^2 + \frac{c_*}{4\delta} \int_\Omega \int_{\tau_1}^{\tau_2} z^2(x,1,s,t) ds dx. \end{aligned} \tag{4.4}$$

By combining (3.4), (4.3) and (4.4), the result follows: Q.E.D.

Theorem 4.3. Assume that (3.2) holds. Then, there exist two positive constants c_3 and c_4 such that

$$E(t) \leq c_3 e^{-c_4 t}.$$

Proof. Setting

$$F_3(t) = F_1(t) + F_2(t).$$

It is easy to verify, for ε small enough, that

$$F_3(t) \sim E(t). \tag{4.5}$$

By using (4.1) and (4.2), we obtain

$$\begin{aligned} F_3'(t) &\leq -(NC_0 - \varepsilon - c_1) \|u_t\|^2 - \varepsilon (1 - \beta - \delta) \|\nabla u\|^2 \\ &-\varepsilon \|\nabla u\|^{2(\gamma+1)} - \left(NC_0 - \varepsilon \frac{c_*}{4\delta} \right) \int_\Omega \int_{\tau_1}^{\tau_2} z^2(x,1,s,t) ds dx \\ &-c_2 \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s(\xi + \mu_2(s)) |z(x,\rho,s,t)|^2 ds d\rho dx. \end{aligned} \tag{4.6}$$

Since $\beta < 1$, choosing δ small enough, such that $\alpha = 1 - \beta - \delta > 0$.

For some $\omega > 0$, the embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ satisfies

$$\begin{aligned} \|u\|_p^p &\leq C \|\nabla u\|_2^p \\ &\leq C \left(\|\nabla u\|^2 \right)^{\frac{p-2}{2}} \|\nabla u\|^2 \\ &\leq C (E(0))^{\frac{p-2}{2}} \|\nabla u\|^2 \\ &\leq \omega \|\nabla u\|^2, \end{aligned}$$

or

$$-\frac{\varepsilon\alpha\omega^{-1}}{2}\|u\|_p^p \geq -\frac{\varepsilon\alpha}{2}\|\nabla u\|_2^2.$$

Hence, (4.6) takes the form

$$\begin{aligned}
F_3'(t) \leq & -(NC_0-\varepsilon-c_1)\|u_t\|^2-\frac{\varepsilon\alpha}{2}\|\nabla u\|^2-\frac{\varepsilon\alpha\omega^{-1}}{2}\|u\|_p^p \\
& -\varepsilon\|\nabla u\|^{2(\gamma+1)}-\left(NC_0-\varepsilon\frac{c_p}{4\delta}\right)\int_\Omega\int_{\tau_1}^{\tau_2} z^2(x,1,s,t)\,dsdx \\
& -c_2\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(\xi+\mu_2(s))|z(x,\rho,s,t)|^2\,dsd\rho dx. \qquad (4.7)
\end{aligned}$$

Whence δ is fixed, choosing N to be large enough, such that

$$NC_0-\varepsilon-c_1>0 \text{ and } NC_0-\varepsilon\frac{c_p}{4\delta}>0.$$

Therefore, (4.7) takes the form, for some $C>0$,

$$\begin{aligned}
F_3'(t) &\leq -C\left[\begin{array}{c}\|u_t\|^2+\|\nabla u\|^2+\|\nabla u\|^{2(\gamma+1)}+\|u\|_p^p \\ +\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(\xi+\mu_2(s))z^2dsd\rho dx\end{array}\right] \\
&\leq -CE(t).
\end{aligned}$$

By using the equivalence relation (4.5) and a simple integration over $(0,t)$, our result proved. Q.E.D.

5 Blow up

In this section, we prove the blow up results for negative initial energy. We have the assumption: $\mu_2:[\tau_1,\,\tau_2]\to R$ is an L^∞ function such that:

$$\left(\frac{2\delta-1}{2}\right)\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds \leq \mu_1,\ \delta>\frac{1}{2}. \qquad (5.1)$$

We have the following lemmas to obtain our main result:

Lemma 5.1. Suppose that (5.1) hold. Let u be a solution of (2.2). Then, $E(t)$ is nonincreasing, so that

$$\begin{aligned}
\mathcal{K}(t) = & \frac{1}{2}\|u_t\|^2+\frac{1}{2}\|\nabla u\|^2+\frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)}+\frac{k}{p^2}\|u\|_p^p \\
& -\frac{1}{p}\int_\Omega|u|^p\ln|u|^k\,dx+\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\left|z^2(x,\rho,s,t)\right|dsd\rho dx, \qquad (5.2)
\end{aligned}$$

which satisfies

$$\mathcal{K}'(t)\leq -c_1\left(\|u_t\|^2+\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|\left|z^2(x,1,s,t)\right|dsdx\right). \qquad (5.3)$$

Proof. By multiplying the first equation of (2.2) by u_t and integrating over Ω , we obtain

$$\begin{aligned} & \frac{d}{dt}\left[\begin{array}{c} \frac{1}{2}\left\|u_t\right\|^2+\frac{1}{2}\left\|\nabla u\right\|^2+\frac{1}{2(\gamma+1)}\left\|\nabla u\right\|^{2(\gamma+1)} \\ +\frac{k}{p^2}\left\|u\right\|_p^p-\frac{1}{p}\int_\Omega \left|u\right|^p \ln\left|u\right|^k dx \end{array}\right] \\ = & -\mu_1\left\|u_t\right\|^2-\int_\Omega u_t\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|\left|z(x,1,s,t)\right| dsdx, \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} & \frac{d}{dt}\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right|\left|z^2(x,\rho,s,t)\right| dsd\rho dx \\ = & -\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} 2\left|\mu_2(s)\right| zz_\rho dsd\rho dx \\ = & \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|\left|z^2(x,0,s,t)\right| dsdx \\ & -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|\left|z^2(x,1,s,t)\right| dsdx \\ = & \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right| ds\right)\left\|u_t\right\|^2 \\ & -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|\left|z^2(x,1,s,t)\right| dsdx. \end{aligned} \tag{5.5}$$

Therefore,

$$\begin{aligned} \frac{d}{dt}\mathcal{K}(t) = & -\mu_1\left\|u_t\right\|^2-\int_\Omega\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|\left|u_t z(x,1,s,t)\right| dsdx \\ & +\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right| ds\right)\left\|u_t\right\|^2 \\ & -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}\left|\mu_2(s)\right|\left|z^2(x,1,s,t)\right| dsdx. \end{aligned} \tag{5.6}$$

From (5.4) and (5.5), we get (5.2). By using Young's inequality, (5.1) and (5.6), we obtain (5.3). As a result, the proof is completed. Q.E.D.

To establish our main result, we define

$$\begin{aligned} H(t) = & -\mathcal{K}(t)=-\frac{1}{2}\left\|u_t\right\|^2-\frac{1}{2}\left\|\nabla u\right\|^2-\frac{1}{2(\gamma+1)}\left\|\nabla u\right\|^{2(\gamma+1)} \\ & -\frac{k}{p^2}\left\|u\right\|_p^p+\frac{1}{p}\int_\Omega\left|u\right|^p\ln\left|u\right|^k dx \\ & -\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu_2(s)\right|\left|z^2(x,\rho,s,t)\right| dsd\rho dx. \end{aligned} \tag{5.7}$$

We give the following lemmas to get our main result:

Lemma 5.2. [12] For $C > 0$,

$$\left(\int_\Omega |u|^p \ln |u|^k \, dx\right)^{s/p} \le C \left[\int_\Omega |u|^p \ln |u|^k \, dx + \|\nabla u\|_2^2\right]$$

satisfies, for any $u \in L^{p+1}(\Omega)$ and $2 \le s \le p$, provided that $\int_\Omega |u|^p \ln |u|^k \, dx \ge 0$.

Lemma 5.3. [12] Depending on Ω only, assume that $C > 0$, so that

$$\|u\|_2^2 \le C \left[\left(\int_\Omega |u|^p \ln |u|^k \, dx\right)^{2/p} + \|\nabla u\|_2^{4/p}\right], \tag{5.8}$$

provided that $\int_\Omega |u|^p \ln |u|^k \, dx \ge 0$.

Lemma 5.4. [12] Depending on Ω only, assume that $C > 0$, such that

$$\|u\|_p^s \le C \left[\|u\|_p^p + \|\nabla u\|_2^2\right], \tag{5.9}$$

for any $u \in L^p(\Omega)$ and $2 \le s \le p$.

Theorem 5.5. Assume that (5.1) holds. Assume further that

$$\begin{cases} p \ge 2, & \text{if } n = 1, 2, \\ 2 < p < \frac{2(n-1)}{n-2}, & \text{if } n \ge 3, \end{cases}$$

and

$$\mathcal{K}(0) < 0. \tag{5.10}$$

Thus, the solution of (2.2) blows up in finite time.

Proof. By (5.3), we know that

$$\mathcal{K}(t) \le \mathcal{K}(0) < 0.$$

Thus,

$$\begin{aligned} H'(t) &= -\mathcal{K}'(t) \\ &\ge c_1 \left(\|u_t\|^2 + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \left|z^2(x,1,s,t)\right| dsdx\right) \\ &\ge c_1 \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \left|z^2(x,1,s,t)\right| dsdx \ge 0 \end{aligned} \tag{5.11}$$

and

$$0 \le H(0) \le H(t) \le \frac{1}{p} \int_\Omega |u|^p \ln |u|^k \, dx. \tag{5.12}$$

We introduce

$$L(t) = H^{1-\alpha}(t) + \varepsilon \int_\Omega u u_t dx + \frac{\mu_1 \varepsilon}{2} \int_\Omega u^2 dx, \ t \ge 0, \tag{5.13}$$

where $\varepsilon > 0$ to be specified later and

$$\frac{2(p-2)}{p^2} < \alpha < \frac{p-2}{2p} < 1. \tag{5.14}$$

By multiplying the first equation in (2.2) by u and with a derivative of (5.13), we get

$$\begin{aligned} L'(t) &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 \\ &\quad + \varepsilon \int_\Omega u u_{tt} dx + \varepsilon \mu_1 \int_\Omega u u_t dx \\ &= (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\nabla u\|^2 - \varepsilon \|\nabla u\|^{2(\gamma+1)} \\ &\quad - \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| u z(x,1,s,t) ds dx + \varepsilon \int_\Omega |u|^p \ln |u|^k dx. \end{aligned} \tag{5.15}$$

Thanks to Young's inequality, we get

$$\begin{aligned} & \varepsilon \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| u z(x,1,s,t) ds dx \\ \leq \; & \varepsilon \left\{ \delta_1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \right) \|u\|^2 \right. \\ & \left. + \frac{1}{4\delta_1} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \left| z^2(x,1,s,t) \right| ds dx \right\}. \end{aligned} \tag{5.16}$$

Therefore, by (5.15), we obtain

$$\begin{aligned} L'(t) &\geq (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 - \varepsilon \|\nabla u\|^2 - \varepsilon \|\nabla u\|^{2(\gamma+1)} \\ &\quad - \varepsilon \delta_1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \right) \|u\|^2 - \frac{\varepsilon}{4\delta_1} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \left| z^2(x,1,s,t) \right| ds dx \\ &\quad + \varepsilon \int_\Omega |u|^p \ln |u|^k dx. \end{aligned} \tag{5.17}$$

By using (5.11) and setting δ_1 such that $\frac{1}{4\delta_1 c_1} = \kappa H^{-\alpha}(t)$, we obtain

$$\begin{aligned} L'(t) &\geq [(1-\alpha) - \varepsilon\kappa] H^{-\alpha}(t) H'(t) + \varepsilon \|u_t\|^2 \\ &\quad - \varepsilon \|\nabla u\|^2 - \varepsilon \|\nabla u\|^{2(\gamma+1)} - \varepsilon \frac{H^\alpha(t)}{4 c_1 \kappa} \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \right) \|u\|^2 \\ &\quad + \varepsilon \int_\Omega |u|^p \ln |u|^k dx. \end{aligned} \tag{5.18}$$

For $0 < a < 1$, we have

$$\begin{aligned} L'(t) \geq\ & [(1-\alpha)-\varepsilon\kappa] H^{-\alpha}(t) H'(t) + \varepsilon a \int_\Omega |u|^p \ln|u|^k dx + \varepsilon \frac{p(1-a)+2}{2} \|u_t\|^2 \\ & + \varepsilon\left(\frac{p(1-a)}{2}-1\right)\|\nabla u\|^2 + \varepsilon\left(\frac{p(1-a)}{2(\gamma+1)}-1\right)\|\nabla u\|^{2(\gamma+1)} \\ & + \frac{\varepsilon(1-a)k}{p}\|u\|_p^p - \varepsilon\frac{H^\alpha(t)}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\|u\|^2 + \varepsilon p(1-a)H(t) \\ & + \frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)||z(x,\rho,s,t)|^2 ds d\rho dx. \end{aligned} \tag{5.19}$$

By using (5.8) and (5.12), we get

$$\begin{aligned} H^\alpha(t)\|u\|_2^2 &\leq \left(\int_\Omega |u|^p \ln|u|^k dx\right)^\alpha \|u\|_2^2 \\ &\leq \left[\left(\int_\Omega |u|^p \ln|u|^k dx\right)^{\alpha+2/p} + \left(\int_\Omega |u|^p \ln|u|^k dx\right)^\alpha \|\nabla u\|_2^{4/p}\right]. \end{aligned}$$

From Young's inequality, we have

$$\begin{aligned} H^\alpha(t)\|u\|_2^2 &\leq \left(\int_\Omega |u|^p \ln|u|^k dx\right)^\alpha \|u\|_2^2 \\ &\leq \left[\begin{array}{c} \left(\int_\Omega |u|^p \ln|u|^k dx\right)^{(p\alpha+2)/p} \\ +\frac{2}{p}\|\nabla u\|^2 + \frac{p-2}{p}\left(\int_\Omega |u|^p \ln|u|^k dx\right)^{\alpha p/(p-2)} \end{array}\right]. \end{aligned}$$

Hence, we get

$$\begin{aligned} H^\alpha(t)\|u\|_2^2 &\leq \left(\int_\Omega |u|^p \ln|u|^k dx\right)^\alpha \|u\|_2^2 \\ &\leq C\left[\begin{array}{c} \left(\int_\Omega |u|^p \ln|u|^k dx\right)^{(p\alpha+2)/p} + \|\nabla u\|^2 \\ +\left(\int_\Omega |u|^p \ln|u|^k dx\right)^{\alpha p/(p-2)} \end{array}\right], \end{aligned}$$

where $C = \max\left\{\frac{2}{p}, \frac{p-2}{p}\right\}$.

By exploiting (5.14), we obtain

$$2 < \alpha p + 2 \leq p \text{ and } 2 < \frac{\alpha p^2}{p-2} \leq p.$$

Thus, lemma 5.2 yields

$$H^\alpha(t)\|u\|_2^2 \leq C\left(\int_\Omega |u|^p \ln|u|^k dx + \|\nabla u\|_2^2\right). \tag{5.20}$$

By combining (5.19) and (5.20), we obtain

$$\begin{aligned} L'(t) &\geq [(1-\alpha)-\varepsilon\kappa] H^{-\alpha}(t) H'(t) \\ &+\varepsilon\left(a-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\right)\int_\Omega |u|^p \ln|u|^k\,dx \\ &+\varepsilon\left(\frac{p(1-a)-2}{2}-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)\right)\|\nabla u\|^2 \\ &+\varepsilon\left(\frac{p(1-a)}{2(\gamma+1)}-1\right)\|\nabla u\|^{2(\gamma+1)} \\ &+\frac{\varepsilon(1-a)k}{p}\|u\|_p^p+\varepsilon\frac{p(1-a)+2}{2}\|u_t\|^2+\varepsilon p(1-a)H(t) \\ &+\frac{\varepsilon p(1-a)}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)||z(x,\rho,s,t)|^2\,ds d\rho dx. \end{aligned} \quad (5.21)$$

Since, choosing $a>0$ so small, such that

$$\frac{p(1-a)-2}{2}>0 \text{ and } \left(\frac{p(1-a)}{2(\gamma+1)}-1\right)>0$$

and by choosing κ large enough, we get

$$\begin{cases} \frac{p(1-a)-2}{2}-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)>0\ , \\ a-\frac{c}{4c_1\kappa}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\right)>0. \end{cases}$$

Once κ and a are fixed, picking ε so small, such that

$$(1-\alpha)-\varepsilon\kappa>0,$$

$$H(0)+\varepsilon\int_\Omega u_0u_1\,dx>0.$$

Thus, for some $\lambda>0$, estimate (5.21) takes the form

$$\begin{aligned} L'(t) &\geq \lambda\Big[H(t)+\|u_t\|^2+\|\nabla u\|^2+\|\nabla u\|^{2(\gamma+1)}+\|u\|_p^p \\ &+\int_\Omega |u|^p\ln|u|^k\,dx+\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)||z(x,\rho,s,t)|^2\,ds d\rho dx\Big], \end{aligned} \quad (5.22)$$

and

$$L(t)\geq L(0)>0,\ t\geq 0. \quad (5.23)$$

From the embedding $\|u\|_2\leq C\|u\|_p$ and Hölder's inequality, we get

$$\int_\Omega uu_t\,dx\leq \|u\|_2\|u_t\|_2\leq C\|u\|_p\|u_t\|_2,$$

then by using Young's inequality, we get

$$\left|\int_{\Omega} uu_t dx\right|^{1/(1-\alpha)} \leq C\left(\|u\|_p^{\mu/(1-\alpha)} + \|u_t\|_2^{\theta/(1-\alpha)}\right), \text{ for } 1/\mu + 1/\theta = 1. \tag{5.24}$$

From Lemma 5.4, we take $\theta = 2(1-\alpha)$ which gives $\mu/(1-\alpha) = 2/(1-2\alpha) \leq p$. Therefore, for $s = 2/(1-2\alpha)$, estimate (5.24) satisfies

$$\left|\int_{\Omega} uu_t dx\right|^{1/(1-\alpha)} \leq C\left(\|u\|_p^{s} + \|u_t\|_2^{2}\right).$$

Therefore, Lemma 5.4 satisfies

$$\left|\int_{\Omega} uu_t dx\right|^{1/(1-\alpha)} \leq C\left[\|\nabla u\|^2 + \|u_t\|^2 + \|u\|_p^p\right]. \tag{5.25}$$

Hence,

$$\begin{aligned}
L^{1/(1-\alpha)}(t) &= \left(H^{1-\alpha}(t) + \varepsilon\int_{\Omega} uu_t dx + \frac{\mu_1\varepsilon}{2}\int_{\Omega} u^2 dx\right)^{1/(1-\alpha)} \\
&\leq C\left[H(t) + \left|\int_{\Omega} uu_t dx\right|^{1/(1-\alpha)} + \|u\|_2^{2/(1-\alpha)}\right] \\
&\leq C\left[H(t) + \left|\int_{\Omega} uu_t dx\right|^{1/(1-\alpha)} + \|u\|_p^{2/(1-\alpha)}\right] \\
&\leq C\left[H(t) + \|\nabla u\|^2 + \|u_t\|^2 + \|u\|_p^p\right], \ t \geq 0.
\end{aligned} \tag{5.26}$$

By combining (5.22) and (5.26), we get

$$L'(t) \geq \Lambda L^{1/(1-\alpha)}(t), \ t \geq 0, \tag{5.27}$$

where Λ is a positive constant depending only on λ and C. A simple integration of (5.27) over $(0, t)$ yields

$$L^{\alpha/(1-\alpha)}(t) \geq \frac{1}{L^{-\alpha/(1-\alpha)}(0) - \Lambda\alpha t/(1-\alpha)}.$$

Therefore, $L(t)$ blows up in time T^*

$$T \leq T^* = \frac{1-\alpha}{\Lambda\alpha L^{\alpha/(1-\alpha)}(0)}.$$

As a result, the proof is completed. Q.E.D.

6 Conclusions

Recently, there has been published much work concerning the wave equations (Kirchhoff, Petrovsky, Bessel,... etc.) with different state of delay time (constant delay, time-varying delay,... etc.). However, to the best of our knowledge, there were no existence, exponential decay and blow up of solutions for the logarithmic Kirchhoff equation with distributed delay. We have been established the global existence, exponential decay and blow up results for the logarithmic Kirchhoff equation with distributed delay under appropriate conditions.

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